The Theory of Lorentz

and

The Principle of Reaction

H. Poincaré

Work of welcome offered by the authors to H.A. Lorentz, Professor of Physics at the University of Leiden, on the occasion of the 25th anniversary of his doctorate, the 11 Dec. 1900.

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It would no doubt seem strange that in a monument raised to the glory of Lorentz I would review the considerations which I presented previously as an objection to his theory. I could say that the pages which follow are rather in the nature of an attenuation rather than a magnification of that objection.

But I disdain that excuse, because I have one which is 100 times better: *Good theories are flexible*. Those which have a rigid form and which can not change that form without collapsing really have too little vitality. But if a theory is solid, then it can be cast in diverse forms, it resists all attacks, and its essential meaning remains unaffected. That's what I discussed at the last Congress of Physics.

Good theories can respond to all objections. Specious arguments have no effect on them, and they also triumph over all serious objections. However, in triumphing they may be transformed.

The objections to them, therefore, far from annihilating them, actually serve them, since they allow such theories to develop all the virtues which were latent in them. The theory of Lorentz is one such, and that is the only excuse which I will invoke.

It is not, therefore, for that for which I will beg the pardon of the reader, but rather for having for so long presented so few novel ideas.

1. To start, let's briefly review the calculation by which one shows that, in the theory of Lorentz, the principle of the equality of action and reaction is not correct, at least if one wishes to apply it solely to material objects.

Let us find the sum of all the ponderable forces applied to all the electrons situated in the interior of a certain volume. That result, or rather its projection on the X axis, is represented by the integral:

$$X = \int \rho\, d\tau \left[\eta\gamma - \zeta\beta + \frac{4\pi f}{K_0}\right],$$

where the integration is taken over all the elements dτ of volume under consideration, and where ξ, η, and ζ represent the components of the velocity of the electron.

From the equations:

$$\rho\eta = -\frac{dg}{dt} + \frac{1}{4\pi}\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right), \qquad \rho\zeta = -\frac{dh}{dt} + \frac{1}{4\pi}\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right),$$

$$\rho = \sum \frac{df}{dx}$$

and adding and subtracting the term $\frac{\alpha}{4\pi}\frac{d\alpha}{dx}$, I can write,

$$X = X_1 + X_2 + X_3 + X_4,$$

where:

$$X_1 = \int d\tau \left(\beta \frac{dh}{dt} - \gamma \frac{dg}{dt}\right),$$

$$X_2 = \int \frac{d\tau}{4\pi}\left(\alpha \frac{d\alpha}{dx} + \beta \frac{d\alpha}{dy} + \gamma \frac{d\alpha}{dz}\right),$$

$$X_3 = \int \frac{-d\tau}{4\pi}\left(\alpha \frac{d\alpha}{dx} + \beta \frac{d\beta}{dx} + \gamma \frac{d\gamma}{dx}\right),$$

$$X_4 = \frac{4\pi}{K_0}\int f\, d\tau \sum \frac{df}{dx}$$

Integration by parts gives us:

$$X_{2'} = \int \frac{d\omega}{4\pi}\alpha(l\alpha + m\beta + \eta\gamma) - \int \frac{d\tau}{4\pi}\alpha\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right),$$

$$X_3 = -\int \frac{d\omega}{8\pi} l(\alpha^2 + \beta^2 + \gamma^2),$$

where the double integrals are taken over all the elements dω of the surface which encloses the volume under consideration, and where *l, m, n* designate the cosines of the angles with the line which is normal to that element.

If we observe that,

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz} = 0,$$

we see that we can write:

$$(1) \qquad X_2+X_3 = \int \frac{d\omega}{8\pi}\left[l\,(\alpha^2-\beta^2-\gamma^2)+2\,m\,\alpha\beta+2\,n\,\alpha\,\gamma\right].$$

We now transform X_4.
Integration by parts gives us:

$$X_4 = \int \frac{4\,\pi\,d\omega}{K_0}(lf^2+mfg+nfh)-\int \frac{4\,\pi\,d\tau}{K_0}\left(f\frac{df}{dx}+g\frac{df}{dy}+h\frac{df}{dz}\right)$$

I call the two integrals on the right hand side X_4' and X_4'', such that

$$X_4 = X'_4 - X''_4 \ .$$

Using the equations,

$$\frac{df}{dy} = \frac{dg}{dx}+\frac{K_0}{4\pi}\frac{d\gamma}{dt},$$
$$\frac{df}{dz} = \frac{dh}{dx}-\frac{K_0}{4\pi}\frac{d\beta}{dt},$$

we can write:

$$X''_4 = Y+Z \ ,$$

where

$$Y = \int \frac{4\pi\,d\tau}{K_0}\left(f\frac{df}{dx}+g\frac{dg}{dx}+h\frac{dh}{dx}\right),$$
$$Z = \int d\tau\left(g\frac{d\gamma}{dt}-h\frac{d\beta}{dt}\right).$$

We then find:

$$Y = \int \frac{2\pi\, l d\omega}{K_0}(f^2 + g^2 + h^2),$$

$$X_1 - Z = \frac{d}{dt}\int d\tau\,(\beta h - \gamma g).$$

And, therefore, we finally have:

$$(2) \qquad X = \frac{d}{dt}\int d\tau\,(\beta h - \gamma) + (X_2 + X_3) + (X'_4 - Y),$$

where $X_2 + X_3$ is given by formula (1), while we have:

$$X'_4 - Y = \int \frac{2\pi\, d\omega}{K_0}[l(f^2 - g^2 - h^2) + 2mfg + 2nfh].$$

The term ($X_2 + X_3$) represents the projection on the *x* axis of the force exerted on the different elements dω of the surface which encloses the volume under consideration. One realizes immediately that the force is none other than the *magnetic pressure* of Maxwell, introduced by that scholar in his well known theory.

In the same way, the term (X'_4 - Y) represents the effect of the *electrostatic pressure* of Maxwell.

Without the presence of the first term:

$$\frac{d}{dt}\int d\tau\,(\beta h - \gamma g),$$

the ponderable force would be none other than that which results from the pressures of Maxwell.

If our integrals are taken over all space, the double integrals in X_2, X_3, X'_4, and Y disappear and all that remains is:

$$X = \frac{d}{dt}\int d\tau\,(\beta h - \gamma g).$$

If, therefore, we call one of the material masses being considered M, and the components of its velocity V_x, V_y, and V_z, and if the principle of reaction were applicable, then we should have:

$$\sum MV_x = \text{const.}, \quad \sum MV_y = \text{const.}, \quad \sum MV_z = \text{const.}$$

On the contrary, we would have:

$$\sum MV_x + \int d\tau(\gamma g - \beta h) = \text{const.},$$
$$\sum MV_y + \int d\tau(\alpha h - \gamma f) = \text{const.},$$
$$\sum MV_z + \int d\tau(\beta f - \alpha g) = \text{const.}$$

Note that

$$\gamma g - \beta h, \qquad \alpha h - \gamma f, \qquad \beta f - \alpha g$$

are the three components of the *radiant vector* of Poynting.

If one defines:

$$\mathrm{J} = \frac{1}{8\pi}\sum \alpha^2 + \frac{2\pi}{K_0}\sum f^2,$$

the Poynting equation then gives us, in fact,

$$(3) \qquad \int \frac{d\mathrm{J}}{dt} d\tau = \int \frac{d\omega}{K_0}\begin{vmatrix} l & m & n \\ \alpha & \beta & \gamma \\ f & g & h \end{vmatrix} + \frac{4\pi}{K_0}\int \rho\, d\tau \sum f\xi$$

The first integral on the right hand side represents, as we know, the quantity of electromagnetic energy which enters the volume under consideration via radiation passing through the surface, and the second term represents the amount of electromagnetic energy which is created within the volume by transformation of other forms of energy.

We can consider the electromagnetic energy as a fictional fluid of which the density is $\mathbf{K_0 J}$ and which travels through space in conformance with Poynting's law. We just need to realize that the fluid is not indestructible, and in the element of volume $d\tau$, during one unit of time, a quantity $\frac{4\pi}{K_0}\rho\, d\tau \sum f\xi$ is destroyed (or, if the sign is negative, a quantity equal to that but with opposite sign is created). It is that which prevents us from considering our fictional fluid as a sort of "real" fluid.

During a unit of time, the amount of the fluid which passes through a surface, equal to **1** and oriented perpendicular to the x axis, or the y axis, or the z axis, is equal to:

$$K_0 \mathrm{J} U_x, \qquad K_0 \mathrm{J} U_y, \qquad K_0 \mathrm{J} U_z,$$

U_x, U_y, U_z being the components of the velocity of the fluid. In comparing this with Poynting's formula, we find:

$$
\begin{aligned}
K_0 \mathrm{J} U_x &= \gamma g - \beta h, \\
K_0 \mathrm{J} U_y &= \alpha h - \gamma f, \\
K_0 \mathrm{J} U_z &= \beta f - \alpha g,
\end{aligned}
$$

and in consequence our formulas become:

$$
(4) \qquad \begin{cases}
\sum MV_x + \int K_0 \mathrm{J} U_x \, d\tau = \text{const.}, \\
\sum MV_y + \int K_0 \mathrm{J} U_y \, d\tau = \text{const.}, \\
\sum MV_z + \int K_0 \mathrm{J} U_z \, d\tau = \text{const.}
\end{cases}
$$

These express the fact that the momentum [literally "quantity of movement"] of the proper matter, plus that of our fictional fluid, is represented by a constant vector.

In ordinary mechanics, if the momentum is constant, then one can conclude that the movement of the center of gravity is linear and uniform.

However, here we do not have the right to conclude that the center of gravity of the system formed by the matter and our fictional fluid is moving linearly and uniformly; and that is because the fluid is not indestructible.

The position of the center of gravity of the fictional fluid is given by the integral

$$\int x \mathrm{J} \, d\tau$$

taken over all space. The derivative of that integral is:

$$\int x \frac{d\mathrm{J}}{dt} d\tau = -\int x d\tau \left(\frac{d\,\mathrm{JU}_x}{dx} + \frac{d\,\mathrm{JU}_y}{dy} + \frac{d\,\mathrm{JU}_z}{dz} \right) - \frac{4\pi}{K_0} \int \rho \, x d\tau \sum f \xi .$$

But the first integral on the right hand side becomes, by integration by parts:

$$\int \mathrm{JU}_x d\tau \quad \text{or} \quad \frac{1}{K_0}(\mathrm{C} - \sum M V_x)$$

where we've used **C** to designate the constant second term of the first of equations (4).

We'll use $\mathbf{M_0}$ to represent the total mass of the matter, we'll use $\mathbf{X_0}$, $\mathbf{Y_0}$, $\mathbf{Z_0}$ for the coordinates of its center of gravity, we'll use $\mathbf{M_1}$ to represent the total mass of the fictional fluid, we'll use $\mathbf{X_1}$, $\mathbf{Y_1}$, $\mathbf{Z_1}$ for the coordinates of its center of gravity, we'll use $\mathbf{M_2}$ for the total mass of the system (matter plus fictional fluid), $\mathbf{X_2}$, $\mathbf{Y_2}$, $\mathbf{Z_2}$ for its center of gravity, and we will then have:

$$M_2 = M_0 + M_1, \qquad M_2 X_2 = M_0 X_0 + M_1 X_1,$$
$$\frac{d}{dt}(M_0 X_0) = \sum M V_x, \qquad K_0 \int x \, \mathrm{J} \, d\tau = M_1 X_1.$$

From that comes:

$$(3) \qquad \frac{d}{dt}(M_2 X_2) = \mathrm{C} - 4\pi \int \rho \, x d\tau \sum f \xi.$$

Here is how one may express equation (3) in ordinary language.

If electromagnetic energy is neither created nor destroyed anywhere, then the last term disappears; then, the center of gravity of the system consisting of the matter and energy (regarded as a fictional fluid) has motion which is linear and uniform.

Let us suppose, now, that at certain locations, there is destruction of electromagnetic energy, which is transformed into non-electrical energy. We must, then, consider the system formed not only by the matter and electromagnetic energy, but also by the non-electrical energy which results from the transformation of the electromagnetic energy.

But we must assume that the non-electrical energy remains at the point where the transformation takes place, and is not subsequently carried along with the matter at that location. There is nothing in this convention which should shock us, as we're only discussing a mathematical fiction. If one adopts that convention, then the movement of the center of gravity of the system will remain linear and uniform.

To extend this statement to the case where there is not only destruction but also creation of energy, it is sufficient to assume that at each point there is a certain quantity of non-electrical energy, from which is formed the electromagnetic energy. One then follows the preceding convention, which is to say that in place of assuming the non-electrical energy is co-located with the ordinary matter, we regard it as immobile. Given that condition, the center of gravity still moves in a straight line.

Consider again equation (2), and suppose the integrals extend over an infinitesimal volume. That signifies, then, that the result of the pressures of Maxwell which act on the surface of the volume under consideration must be in equilibrium:

1. With the non-electrical forces which are acting on the matter which is situated within the volume;
2. With the inertial forces of that matter;
3. With the inertial forces of the fictional fluid enclosed in the volume.

To define the inertia of that fictional fluid, we must assume that the fluid which is created at any point by transformation of non-electrical energy is created without velocity and that it obtains its velocity from the fluid which already exists. If, therefore, the quantity of fluid increases, but the velocity remains constant, we must have a certain inertia to overcome since the new fluid "borrows" its velocity from the old fluid. The velocity of the system would diminish if some cause did not intervene to keep it constant. Similarly, when there is destruction of electromagnetic energy, the fluid which is destroyed must lose its velocity prior to its destruction by giving it up to the remaining fluid.

If equilibrium holds for an infinitesimal volume, it must also hold for a finite volume. If, in fact, we decompose it into infinitesimal volumes, the equilibrium holds for each of them. To pass to a finite volume, we must consider the collection of forces applied to different infinitesimal volumes; among the pressures of Maxwell we retain only those which act on the finite total surface of the volume, but ignore those which act on those surface elements which separate two contiguous infinitesimal volumes. That does not affect the equilibrium, since the pressures we're ignoring are pairwise equal and opposite.

The equilibrium would therefore hold for finite volumes.

It would therefore hold for all of space. However, in that case, we consider neither the pressures of Maxwell which are zero at infinity, nor the non-electrical forces which are in equilibrium by virtue of the principle of reaction of forces of ordinary mechanics.

The two types of inertial forces are therefore in equilibrium, from which we have a double consequence:

1. The principle of conservation of the projections of the momenta apply to the system of matter and of fictional fluid. We can also derive the equations (4).

2. The principle of conservation of the moments of the momenta or, in other terms, *conservation of angular momentum* [literally “principle of areas”] applies to the system of matter and fictional fluid. This is a new consequence, which completes the information gleaned from equations (4).

Therefore, from our point of view, since the electromagnetic energy behaves as a fluid which has inertia, we must conclude that, if any sort of device produces electromagnetic energy and radiates it in a particular direction, that device must *recoil* just as a cannon does when it fires a projectile.

Of course, that recoil won't take place if the device emits energy equally in all directions; it will only take place if the emission is asymmetric, and if the electromagnetic energy is emitted in a single direction, such as happens, for example, if the device is a Hertzian exciter placed at the focus of a parabolic mirror.

It is easy to evaluate that recoil quantitatively. If the device has a mass of 1 kg and if it emits three million joules in one direction with the velocity of light, the speed of the recoil is 1 cm/sec. In other terms, if the energy produced by a machine of 3,000 watts is emitted in a single direction, a force of one dyne is needed in order to hold the machine in place despite the recoil.

It is apparent that such a weak force couldn't be detected in our experience. But we can imagine that, impossibly, we have measuring devices so sensitive that we can measure such forces. We could then demonstrate that the principle of reaction is applicable not just to matter; and that would be confirmation of the theory of Lorentz, and the downfall of some other theories.

But that is nothing; the theory of Hertz and, in general, all the other theories predict the same recoil as the theory of Lorentz.

I've already considered the example of a Hertzian exciter of which the radiation is rendered parallel by use of a parabolic mirror. I will now consider a simpler example, borrowed from optics: a parallel bundle of light rays striking a mirror perpendicularly, reversing their direction upon reflection. The energy initially traveling from left to right, for example, is subsequently returned from right to left by the mirror.

The mirror must therefore *recoil* and the recoil is easy to calculate using our previous considerations.

But it's easy to recognize the problem which has already been treated by Maxwell in paragraphs 792 and 793 of his Works. He also predicts a recoil of the mirror exactly the same as that which we've deduced from the theory of Lorentz.

If, in fact, we go farther into the study of the mechanism of the recoil, here is what we find. Consider some volume, and apply equation (2); that equation tells us that the electromagnetic force which is exerted on the electrons, which is to say on the matter contained in the volume, is equal to the resultant of the pressures of Maxwell augmented by a correction term which is the derivative of the integral

$$\int d\tau(\beta h - \gamma g).$$

If the situation is static [literally “established”], then this integral is constant and the correction term is zero.

The recoil predicted by the theory of Lorentz is that which is due to the pressures of Maxwell. But all the theories predict the pressures of Maxwell; *therefore, all the theories predict the same recoil.*

2. But then a question arises. We have predicted the recoil using the theory of Lorentz because that theory is contrary to the principle of reaction. But among the other theories, there are those, such as that of Hertz, which conform to that principle. How can they all lead to the same recoil?

I hasten to give the resolution to that paradox, which I'll justify later on. In the theory of Lorentz and in that of Hertz, the device which produces the energy and emits it unidirectionally recoils, but the energy so radiated propagates through a certain medium; the air, for example.

In the theory of Lorentz, when the air receives the energy so radiated, it does not result in any mechanical action; it also is unaffected when the energy leaves it after traversing it. In contrast, in the theory of Hertz, when the air receives the energy, it is pushed forward, and it recoils back when the energy leaves it. The movements of the air traversed by the energy thus compensate, from the point of view of the principle of reaction, the movements of the device which produced the energy. In the theory of Lorentz, that compensation does not happen.

Let us look again at the theory of Lorentz and at our equation (2), and apply it to a homogeneous dielectric. We know how Lorentz represents a dielectric material; that medium contains some electrons which are susceptible to small displacements, and those displacements produce the dielectric polarization, the effect of which, from certain points of view, is then added to the proper electrical displacement.

Let X, Y, Z be the components of that polarization. We would then have:

$$(5) \qquad \frac{dX}{dt} d\tau = \sum \rho \xi, \qquad \frac{dY}{dt} d\tau = \sum \rho \eta, \qquad \frac{dZ}{dt} d\tau = \sum \rho \zeta \quad .$$

The summations of the right hand sides of equations (5) extend over all the electrons contained in the interior of the volume element $d\tau$, and these equations could be regarded as the definition itself of the dielectric polarization.

For the expression of the resultant of the ponderable forces (which I no longer call "X", in order to avoid confusion with the polarization), we have found the integral:

$$\int \rho \, d\tau \left[\eta \gamma - \zeta \beta + \frac{4\pi f}{K_0} \right]$$

or

$$\int \rho \eta \gamma \, d\tau - \int \rho \zeta \beta \, d\tau + \frac{4\pi}{K_0} \int \rho f \, d\tau \quad .$$

The first two integrals can be replaced by

$$\int \gamma \frac{dX}{dt} d\tau , \qquad \int \beta \frac{dZ}{dt} d\tau$$

by virtue of the equations (5). As for the third integral, it's zero, because the net charge of an element of the dielectric which contains a particular number of electrons is zero. Therefore, our ponderable force reduces to:

$$\int \left(\gamma \frac{dY}{dt} - \beta \frac{dZ}{dt}\right) d\tau .$$

If, then, I call the force due to the various pressures of Maxwell "**II**", such that

$$\mathrm{II} = (X_2 + X_3) + (X'_4 - Y),$$

then our equation (2) becomes:

$$\text{(2 } bis\text{)} \qquad \mathrm{II} = \int \left(\gamma \frac{dY}{dt} - \beta \frac{dZ}{dt}\right) d\tau + \frac{d}{dt}\int (\gamma g - \beta h)\, d\tau .$$

We also have a relationship such as this

$$\text{(A)} \qquad a\frac{d^2 X}{dt^2} + bX = f ,$$

where a and b are two constants characteristic of the medium; from that, we can easily deduce:

$$\text{(B)} \qquad X = (n^2 - 1) f$$

and, in the same way,

$$Y = (n^2 - 1) g , \qquad Z = (n^2 - 1) h ,$$

n being the index of refraction of the color under consideration.

We could be tempted to replace the relationship (A) with others which are more complicated; for example, if we should consider complex ions. It would make little difference, as we would still arrive at equation (B).

To carry this farther, as an example, we are going to suppose a plane wave is propagated along the x axis in the positive direction. If the wave is polarized in the xz plane, we would have,

$$X = f = g = \alpha = Z = h = \beta = 0$$

and

$$\gamma = ng \frac{4\pi}{\sqrt{K_0}}.$$

Taking account of all these relationships, (2 *bis*) becomes, to start with,

$$\mathrm{II} = \int \gamma \frac{dY}{dt} d\tau + \int \gamma \frac{dg}{dt} d\tau + \int g \frac{d\gamma}{dt} d\tau,$$

where the first integral represents the ponderable force. But if we take account of the proportions

$$\frac{g}{1} = \frac{Y}{n^2-1} = \frac{\gamma}{n\left(\frac{4\pi}{\sqrt{K_0}}\right)},$$

our equation becomes

$$(6) \qquad \frac{\sqrt{K_0}}{4\pi}\mathrm{II} = n(n^2-1)\int g \frac{dg}{dt} d\tau + n\int g \frac{dg}{dt} d\tau + n\int g \frac{dg}{dt} d\tau.$$

But to make something of that formula, it is valuable to see how the energy divides up and propagates in a dielectric material. The energy divides into three parts: 1st, the electrical energy; 2nd, the magnetic energy; 3rd, the mechanical energy due to movement of the ions. The expressions for these three parts are, respectively,

$$\frac{2\pi}{K_0}\sum f^2, \quad \frac{1}{8\pi}\sum \alpha^2, \quad \frac{2\pi}{K_0}\sum f X$$

and in the case of a plane wave, these are in the ratios,

$$1, \quad n^2, \quad n^2-1.$$

In the preceding analysis, we have given a role to that which we have called the momentum of the electromagnetic energy. It is clear that the density of our fictional fluid will be proportional to the sum of the first two parts (electrical and magnetic) of the total energy, and that the third part, which is purely mechanical, should be left to one side. But what speed should we attribute to the fluid? To start, one might think it should be the speed of propagation of the wave, which is to say $\frac{1}{n\sqrt{K_0}}$. However, it's not that simple. At each point, there is a proportionality between the electromagnetic and mechanical energy; if, therefore, at one point the electromagnetic energy were to diminish, the mechanical energy would diminish equally, which is to say part of it transforms into electromagnetic energy; there would be, therefore, creation of fictional fluid at that point.

For the moment, let us designate the density of the fictional fluid by ρ, and its velocity by ξ, which I will assume is parallel to the *x* axis. I will assume that all our functions depend only on *x* and *t*, the plane of the wave being perpendicular to the *x* axis. The continuity equation is then written as:

$$\frac{d\rho}{dt} + \frac{d\rho\,\xi}{dx} = \frac{\delta\rho}{dt},$$

where δρ is the quantity of fictional fluid created during the time *dt*. But that quantity is equal to the amount of mechanical energy destroyed, which is to the amount of electromagnetic energy destroyed, $-d\rho$, as $n^2 - 1$ is to $n^2 + 1$; from which

$$\frac{\delta\rho}{n^2-1} = -\frac{d\rho}{n^2+1},$$

from which our equation becomes

$$\frac{d\rho}{dt}\frac{2\mathrm{n}^2}{n^2+1} + \frac{d\rho\,\xi}{dx} = 0.$$

If ξ is a constant, that equation shows us that the speed of propagation is equal to

$$\xi\frac{n^2+1}{2\mathrm{n}^2}.$$

If the speed of propagation were $\frac{1}{n\sqrt{K_0}}$, we would therefore have

$$\xi = \frac{2\mathrm{n}}{(n^2+1)\sqrt{K_0}}.$$

If the total energy is J', the electromagnetic energy will be $\mathrm{J} = \frac{n^2+1}{2n^2}\mathrm{J}'$ and the momentum of the fictional fluid will be:

$$(7) \qquad K_0 \mathrm{J}\,\xi \;=\; K_0 \frac{n^2+1}{2\,n^2}\mathrm{J}'\xi \;=\; \frac{\mathrm{J}'\sqrt{K_0}}{n}$$

since the density of the fictional fluid is equal to the energy multiplied by K_0.
But in equation (6) the first term of the right hand side represents the ponderable force, which is to say the derivative of the momentum of the dielectric material, while the two last terms represent the derivative of the momentum of the fictional fluid. These two momenta are therefore in the ratio of n^2-1 to 2.

Let the density of the dielectric material be Δ, and the components of its velocity be W_x, W_y, and W_z. Recall the equations (4). The first term, $\sum MV_x$ represents the motion of all the real matter; we'll decompose it into two parts. The first part, which we'll continue to designate as $\sum MV_x$, will represent the momentum of the device producing the energy. The second part will represent the momentum of the dielectrics. It will be equal to

$$\int \Delta . W_x d\tau$$

from which equation (4) will become

$$(4\ \textit{bis}) \qquad \sum MV_x + \int (\Delta . W_x + K_0 \mathrm{J}\, U_x)\, d\tau \;=\; \text{const.}$$

From that which we have just seen, we have

$$\frac{\Delta . W_x}{n^2-1} \;=\; \frac{K_0 \mathrm{J}\, U_x}{2}.$$

Furthermore, call the total energy, as above, J'. We will also distinguish the real velocity of the fictitious fluid, which is to say that which results from Poynting's law, and which we have designated as U_x, U_y, U_z, from the apparent speed of the energy, which is to say that which we deduced from the propagation velocity of the waves and which we designated as U'_x, U'_y, U'_z. From equation (7), we obtain:

$$\mathrm{J}\,U_x = \mathrm{J}'U'_x.$$

We can, therefore, write equation (4 *bis*) in the form:

$$\sum MV_x + \int (\Delta . W_x + K_0 \mathrm{J}'U'_x) d\tau = \text{const.}$$

Equation (4 *bis*) shows the following: If a device radiates energy in a single direction *in vacuum*, it undergoes a recoil which, from the point of view of the principle of reaction, is compensated solely by the movement of the fictitious fluid.

But if, instead, the radiation takes place in a dielectric, the recoil will be compensated partly by the movement of the fictitious fluid and partly by the movement of the dielectric material, and the fraction of the recoil of the device which will thus be compensated by the movement of the dielectric, which is to say by the movement of some real matter, I would say, will be $\frac{n^2-1}{n^2+1}$.

That was what resulted from the theory of Lorentz. We will now pass on to the theory of Hertz.

We know what the constitution of a dielectric is according to the ideas of Mossotti.

Dielectrics other than the vacuum would be formed of tiny conductive spheres (or more generally, of tiny conductive bodies) separated from each other by an insulating and non-polarizable medium which is analogous to the vacuum. How can we pass from there to the ideas of Maxwell? We imagine that the vacuum itself has the same structure; it is not non-polarizable, but formed of conductive cells, separated by partitions formed of an ideal material, insulating and non-polarizable. The specific inductance of the vacuum would thus be larger than that of an ideal non-polarizable material (just as in the primitive concepts of Mossotti, the inductance of a dielectric is larger than that of the vacuum, and for the same reason). And the inductance of the vacuum would be increased relative to that of the ideal material, as the space occupied by the conductive cells is increased relative to the space occupied by the insulating partitions.

In the limit, we regard the inductance of the insulating material as infinitely small, and at the same time the insulating partitions as infinitely thin, in such fashion that the space occupied by the partitions being infinitely small, the inductance of the vacuum remains finite. *That passage to the limit brings us to the theory of* Maxwell.

All of this is well known and I'll restrict myself to a brief review. Note that *there is the same relationship between the theory of* Lorentz *and that of* Hertz *as there is between that of* Mossotti *and that of* Maxwell.

Let us suppose, in fact, that we attribute to the vacuum the same constitution as Lorentz attributes to ordinary dielectrics; that is to say that we consider it as a non-polarizable medium in which some electrons may be subject to small displacements.

The formulas of Lorentz will still be applicable, *only* K_0 *no longer represents the inductance of the vacuum, but that of our ideal non-polarizable medium.* Let us pass to the limit by supposing K_0 is infinitely small; we emphasize that to compensate for that hypothesis, we multiply the number of electrons so that the inductance of the vacuum and the other dielectrics remain finite.

The theory at which we arrive at the limit is none other than that of Hertz.

Let V be the speed of light in vacuum. In the primitive theory of Lorentz, it is equal to $\frac{1}{\sqrt{K_0}}$; but it is no longer the same in the modified theory, where it is equal to

$$\frac{1}{n_0\sqrt{K_0}}$$

n_0 being the index of refraction of the vacuum relative to an ideal non-polarizable medium. If n is the index of refraction of a dielectric relative to common vacuum, then its index relative to that ideal medium will be nn_0 and the speed of light in that dielectric will be

$$\frac{V}{n} = \frac{1}{n n_0\sqrt{K_0}}.$$

In the formulas of Lorentz, we must replace n with nn_0.

For example, the dragging of the waves in Lorentz's theory is represented by Fresnel's formula

$$v\left(1-\frac{1}{n^2}\right).$$

In the modified theory, it would be

$$v\left(1-\frac{1}{n^2 n_0^2}\right).$$

If we pass to the limit, we must set $K_0 = 0$, from which we have $n_0 = \infty$; therefore, in the theory of Hertz, the dragging velocity will be v, which is to say the dragging will be complete. That consequence, which is contrary to the result of Fizeau, is sufficient to condemn the theory of Hertz, such that we consider it as little more than a curiosity.

Let us consider again our equation (4 *bis*). It tells us that the fraction of the recoil which is compensated by the movement of the dielectric material is equal to

$$\frac{n^2-1}{n^2+1}\ .$$

In the modified theory of Lorentz, that fraction will be:

$$\frac{n^2 n_0^2-1}{n^2 n_0^2+1}$$

If we pass to the limit by making $n_0 = \infty$, that fraction is equal to 1, and consequently the recoil is *entirely* compensated by the movement of the dielectric material. In other words, in Hertz's theory, the principle of reaction isn't violated, and applies solely to matter.

We can also see that from equation (4 *bis*); if, in the limit, K_0 is zero, the term $\int K_0 J' U'_x d\tau$ which represents the motion of the fictional fluid also goes to zero; consequently it is sufficient to consider the motion of the real matter.

From which we have this consequence: *To demonstrate experimentally that the principle of reaction is indeed violated in reality, as it is in* Lorentz's *theory, it is not sufficient to show that the device producing the energy recoils,* which would already be very difficult, *it is necessary to also show that the recoil is not compensated by the movement of the dielectric, and in particular the motion of the air traversed by the electromagnetic waves.* That would clearly be still far more difficult.

One last remark on this subject. Suppose that the medium traversed by the waves is magnetic. A part of the energy of the wave will still take the form of mechanical energy. If μ is the magnetic permeability of the medium, then the *total* magnetic energy will be:

$$\frac{\mu}{8\pi}\int \sum \alpha^2 d\tau\ ,$$

but a fraction only, specifically:

$$\frac{1}{8\pi}\int \sum \alpha^2 d\tau$$

will be properly called magnetic energy; the other part:

$$\frac{\mu-1}{8\pi}\int\sum\alpha^2\,d\tau$$

will be the *mechanical* energy used to bring the particular currents into a common orientation perpendicular to the field, against the elastic force which tends to move the currents into the equilibrium orientation which they'd take in the absence of a magnetic field.

We could, therefore, apply an analysis to this medium which is altogether parallel to the preceding analysis, and where the mechanical energy $\frac{\mu-1}{8\pi}\int\sum\alpha^2\,d\tau$, plays the same role played by the mechanical energy $\frac{2\pi}{K_0}\int\sum Xf d\tau$ in the case of a dielectric. Thus, we can see that, if there are magnetic media which are not dielectric (I mean, in which the dielectric property would be the same as that of the vacuum), the material of these media must undergo a mechanical action due to the passage of the waves such that the recoil of the device is partly compensated by the movements of the media, just as it is by dielectrics.

To move away from that unrealistic case, if we assume a medium which is at once dielectric and magnetic, the fraction of the recoil compensated by the movement of the medium will be larger than for a medium which is not magnetic but which is equally dielectric.

3. Why does the principle of reaction matter to us? It is important to consider this question, to see if the paradoxes we've discussed can really be considered as an objection to Lorentz's theory.

If that principle, in most cases, imposes itself on us, it's because its negation leads to perpetual motion. Is that also the case here?

Let A and B be two bodies of any sort, with one acting on the other, isolated from all external forces; if the action of one weren't equal to the reaction of the other, we could connect them with a rod of fixed length such that they behaved as *a single* solid body. The forces applied to that solid not being in equilibrium, the system would cause itself to move and that motion would accelerate continuously, *for all time*, as the interaction of the two bodies depends only on their *relative* positions and their *relative* velocities, but is independent of their *absolute* position and their *absolute* velocities.

More generally, given a conservative system of any sort, where U is its potential energy, *m* the mass of one of the points of the system, and x', y', z' are the components of its velocity, we would have the energy equation [literally "equation of lively forces"]:

$$\sum\frac{m}{2}(x'^2+y'^2+z'^2)+U=constant$$

Now let us move to a coordinate system which is moving with constant velocity *v* parallel to the x axis. Letting x', y', z' be the components of the velocity relative to these axes, we have:

$$x' = x'_1 + v, \quad y' = y'_1, \quad z' = z'_1,$$

and, consequently:

$$\sum \frac{m}{2}\left[(x'_1+v)^2 + y'^2_1 + z'^2_1\right] + U = \text{const.}$$

By virtue of the *principle of relative motion*, U depends only on the *relative* position of the points of the system, the laws of relative motion being no different from the laws of absolute motion, and the energy equation for relative motion is written as:

$$\sum \frac{m}{2}(x'^2_1 + y'^2_1 + z'^2_1) + U = \text{constant}$$

Separating the two equations, we find

(8) $$v \sum mx'_1 + \frac{v^2}{2} \sum m = \text{constant}$$

or

(9) $$\sum mx'_1 = \text{constant}$$

which is the analytic expression of the principle of reaction.

The principle of reaction appears to us, therefore, as a consequence of the principle of energy and the principle of relativity of motion. The latter weighs heavily on our thoughts when we consider an isolated system.

But in the case which we're considering, we're not dealing with an isolated system, since we're only considering the ordinary matter, and in addition to that there is still an ether. If all material objects are carried along by a common translation, as, for example, the motion of the Earth, phenomena could be different from those which we would observe in the absence of that translation since the ether could not be carried along by the translation. It seems like the principle of relativity of motion should not just apply to ordinary matter; so, experiments have been carried out to detect the motion of the Earth. Those experiments, it is true, have produced negative results, but we find that rather astonishing.

All the same a question remains. Those experiments, as I said, produced a negative result, and the theory of Lorentz explains that negative result. It appears that the principle of relativity of motion, which is not clearly true *a priori*, is verified *a posteriori* and that the principle of reaction should follow. Yet the principle of reaction does not hold; how can that be?

It is the case that, in reality, that which we call the principle of relativity of motion has been verified only imperfectly, as shown by the theory of Lorentz. This is due to the compensation of multiple effects, but:

1. That compensation does not take place unless we neglect v^2, at least as regards a certain complementary hypothesis which I won't discuss for the moment.

All the same that is not important to our subject, because if we neglect v^2, equation (8) leads directly to equation (9), which is to say the principle of reaction.

2. For the compensation to work, we must relate the phenomena not to the true time t, but to a certain *local time* t' defined in the following fashion.

Let us suppose that there are some observers placed at various points, and they synchronize their clocks using light signals. They attempt to adjust the measured transmission time of the signals, but they are not aware of their common motion, and consequently believe that the signals travel equally fast in both directions. They perform observations of crossing signals, one traveling from A to B, followed by another traveling from B to A. The local time t is the time indicated by the clocks which are so adjusted.

If $V=\frac{1}{\sqrt{K_0}}$ is the speed of light, and v is the speed of the Earth which we suppose is parallel to the x axis, and in the positive direction, then we have:

$$t'=t-\frac{v\,x}{V^2}$$

3. In relative motion, the propagation of the apparent energy follows the same laws as the real energy in absolute motion, but the apparent energy is not exactly equal to the corresponding real energy.

4. In relative motion, the bodies emitting the electromagnetic energy are subject to an apparent complementary force which does not exist in absolute motion.

We will see how these several circumstances resolve the contradiction which we pointed out above.

Let us imagine that a device produces electrical energy, such that the energy is emitted in a single direction. That could be, for example, a Hertzian exciter equipped with a parabolic mirror.

Initially at rest, the exciter emits some energy along the *x* axis, and that energy is exactly equal to that which is expended by the exciter. As we have seen, the device *recoils* and takes on a certain velocity.

If we relate everything to the moving axes which are linked to the exciter, the apparent phenomena should be, except for the reservations mentioned above, the same as if the exciter were at rest; it will therefore radiate an *apparent* quantity of energy which is equal to the energy expended in the exciter.

On the other hand, it receives an impulse from the recoil, and since it is not stopped, but already has a nonzero velocity, that impulse does work on the device and its kinetic energy increases.

If, therefore, the *real* electromagnetic energy radiated by the device were equal to the apparent electromagnetic energy, which is, as I just said, equal to the energy expended in the exciter, the increase in kinetic energy of the device would be obtained without any corresponding consumption. That is contrary to the principle of conservation. If, therefore, it undergoes recoil, the apparent energy must not be equal to the real energy and the phenomena in relative motion will not be exactly the same as those in absolute motion.

Let us examine this a little more closely. Suppose v' is the speed of the exciter, v is the speed of the moving coordinates, which we'll no longer assume are linked to the exciter, and V is the speed of the radiation. All velocities are assumed to be parallel to the *x* axis, and positive. To simplify it, let us suppose that the radiation has the form of a polarized plane wave, for which we have the equations:

$$f = h = \alpha = \beta = 0 \ ,$$

$$4\pi \frac{dg}{dt} = -\frac{d\gamma}{dx}, \qquad -\frac{1}{4\pi V^2}\frac{d\gamma}{dt} = \frac{dg}{dx}, \qquad V\frac{d\gamma}{dx} + \frac{d\gamma}{dt} = 0\,,\,,$$

from which we obtain:

$$\gamma = 4\pi V g \, .$$

The real energy contained within the volume will be:

$$\frac{\gamma^2}{8\pi} + 2\pi V^2 g^2 \ = \ 4\pi V^2 g^2 \, .$$

Let us now see what the apparent motion is relative to the moving axes. For the apparent electric and magnetic fields, we have:

$$g' = g - \frac{v}{4\pi V^2}\gamma, \qquad \gamma' = \gamma - 4\pi v g.$$

We have, therefore, for the apparent energy in the volume under consideration (neglecting v^2 but not vv'):

$$\frac{\gamma'^2}{8\pi} + 2\pi V^2 g'^2 = \left(\frac{\gamma^2}{8\pi} - v g \gamma\right) + 2\pi V^2\left(g^2 - \frac{v g \gamma}{2\pi V^2}\right)$$

or

$$4\pi V^2 g^2 - 2 v g \gamma = 4\pi V^2 g^2\left(1 - \frac{2v}{V}\right).$$

Furthermore, the apparent equations of motion can be written,

$$4\pi\frac{dg'}{dt'} = -\frac{d\gamma'}{dx'}, \qquad -\frac{1}{4\pi V^2}\frac{d\gamma'}{dt'} = \frac{dg'}{dx'},$$

which shows that the apparent propagation velocity is still V.

Suppose T is the duration of the emission; what will the real length in space be of the perturbation?

The head of the perturbation exits the device at time 0 at location 0, and at time t it will be at point Vt. The tail exits the device at time T, but not at point 0, rather at point v'T, since the exciter from which it emerges moved with a velocity v' during the time interval T. At time t, the tail is therefore at location $v'T + V(t - T)$. The real length of the perturbation is therefore

$$L = Vt - [v'T + V(t - T)] = (V - v')T.$$

Now, what is the apparent length? The head emerges at local time 0, at local coordinate 0; at local time t' the coordinate relative to the moving axes will be Vt'. The tail emerges at time T at point v'T of which the coordinate relative to the moving axis is $(v' - v)$T; the local time corresponding to that is

$$T\left(1 - \frac{vv'}{V^2}\right).$$

At local time t', it is at point x, where x is given by the equations:

$$t' = t - \frac{vx}{V^2}, \qquad x = v'T + V(t - T) \ ,$$

from which, neglecting v^2 :

$$x = [v'T + V(t' - T)]\left(1 + \frac{v}{V}\right) .$$

The x coordinate of that point relative to the moving axes will be

$$x - vt' = (v'T - VT)\left(1 + \frac{v}{V}\right) + Vt' .$$

The apparent length of the perturbation will be, therefore,

$$\mathrm{L}' = \mathrm{V}t' - (x - vt') = (\mathrm{V} - v')\mathrm{T}\left(1 + \frac{v}{\mathrm{V}}\right) = \mathrm{L}\left(1 + \frac{v}{\mathrm{V}}\right)$$

The total real energy (per unit section) is therefore

$$\left(\frac{\gamma^2}{8\pi} + 2\pi \mathrm{V}^2 g^2\right)\mathrm{L} = 4\pi \mathrm{V}^2 g^2 \mathrm{L} \ ,$$

and the apparent energy

$$\left(\frac{\gamma'^2}{8\pi} + 2\pi \mathrm{V}^2 g'^2\right)\mathrm{L}' = 4\pi \mathrm{V}^2 g^2 \mathrm{L}\left(1 - \frac{2\mathrm{v}}{\mathrm{V}}\right)\left(1 + \frac{v}{\mathrm{V}}\right) = 4\pi \mathrm{V}^2 g^2 \mathrm{L}\left(1 - \frac{v}{\mathrm{V}}\right) .$$

If $\mathrm{J}dt$ represents the real radiated energy during time dt, then $\mathrm{J}dt\left(1 - \frac{v}{\mathrm{V}}\right)$ will represent the apparent energy.

Suppose $\mathrm{D}dt$ is the energy expended in the exciter; it is the same in absolute motion and in apparent motion.

We still must account for the recoil. The force of recoil, multiplied by *dt,* is equal to the increase in the momentum of the fictitious fluid, which is to say,

$$dt\,\mathrm{K}_0\mathrm{J}\,\mathrm{V} \;=\; \frac{\mathrm{J}}{\mathrm{V}}\,dt\;,$$

since the amount of fluid which is created is $\mathrm{K}_0\mathrm{J}\,dt$ and its velocity is V. The work of recoil is therefore:

$$-\frac{v'\,\mathrm{J}\,dt}{\mathrm{V}}\;.$$

In the case of apparent motion, we must replace v' with v'-v and J with $\mathrm{J}\left(1-\dfrac{v}{\mathrm{V}}\right)$.

The apparent work of recoil is therefore:

$$-\frac{(v'-v)\,\mathrm{J}\,dt}{\mathrm{V}}\left(1-\frac{v}{\mathrm{V}}\right) \;=\; \mathrm{J}\,dt\left(-\frac{v'}{\mathrm{V}}+\frac{v}{\mathrm{V}}+\frac{vv'}{\mathrm{V}^2}\right)\;.$$

Finally, in apparent motion, we must account for the apparent complementary force of which I spoke above (4.). That complementary force is equal to $-\dfrac{v\,\mathrm{J}}{\mathrm{V}^2}$ and the work it does, neglecting v^2, is $-\dfrac{vv'}{\mathrm{V}^2}\mathrm{J}\,d\tau$.

Given that, the equation of the kinetic energy in real motion is:

$$\mathrm{J}-\mathrm{D}-\frac{v'\mathrm{J}}{\mathrm{V}} \;=\; 0\;. \tag{10}$$

The first term represents the radiated energy, the second the expended energy and the third the work of recoil.

The equation of the kinetic energy in apparent motion is:

$$(11) \qquad J\left(1-\frac{v}{V}\right)-D+J\left(-\frac{v'}{V}+\frac{v}{V}+\frac{vv'}{V^2}\right)-\frac{vv'}{V^2}J = 0 .$$

The first term represents the apparent radiated energy, the second the energy expended, the third the apparent work of recoil, and the fourth the work of the apparent complementary force.

The correspondence between equations (10) and (11) removes the apparent contradiction which I pointed out above.

If, therefore, in Lorentz's theory, the recoil can take place without violating the energy principle, it's because the apparent energy as measured by an observer carried along with the moving axes is not equal to the real energy. Let us suppose, then, that our exciter recoils and that the observer is carried along with that motion ($v' = v < 0$). The exciter would appear immobile to that observer, and to him it would appear that the radiated energy was equal to the energy radiated by an exciter at rest. But in reality it radiated less, and that difference is what makes up for the work of recoil.

I could have assumed the moving axes were permanently linked to the exciter, which is to say $v = v'$, but my analysis would not then have demonstrated the role of the apparent complementary force. To do that I must assume v' is much larger than v, so that I can neglect v^2 but not vv'.

I could also show the necessity for the apparent complementary force in the following fashion:

The real recoil is $\frac{J}{V}$; in apparent motion, we must replace J with $J\left(1-\frac{v}{V}\right)$ such that the apparent recoil is

$$\frac{J}{V}-\frac{Jv}{V^2}$$

To complete the expression for the real recoil, we must therefore add to the apparent recoil an apparent complementary force $\frac{Jv}{V^2}$ (I make the sign − because the recoil, as indicated by its name, takes place in the negative direction).

The existence of the apparent complementary force is therefore a necessary consequence of the phenomenon of recoil.

Thus, according to the theory of Lorentz, the principle of reaction must not apply solely to matter; the principle of relative motion also must not apply solely to matter. It's important to note that there is an intimate and necessary connection between these two facts.

It is, therefore, sufficient to establish experimentally either of the two, from which the other will be established, ipso facto. It would no doubt be less difficult to demonstrate the second; but even that is nearly impossible since, for example, Liénard has calculated that with a machine of 100 kW, the apparent complementary force would be no more than $\frac{1}{600}$dyne.

An important consequence stems from the correlation of the two facts. That is that the Fizeau experiment itself is already contrary to the principle of reaction. If, in fact, as indicated by that experiment, the waves are dragged only partially, then the *relative* propagation of the waves in a moving medium must not follow the same laws as propagation in a stationary medium, which is to say the principle of relative motion does not apply solely to matter, and we must make at least one additional correction, of which I spoke above (2.) and which consists of relating everything to "local time". If that correction is not compensated by others, we must conclude that the principle of reaction is not correct for matter alone.

Thus, all theories which respect that principle are condemned, *at least unless we consent to profoundly modify all our ideas regarding electrodynamics.* That is an idea which I have developed at greater length in an earlier article (*l'Éclairage Électrique*, volume 5, no. 40, page 395).

Translator's Note:

The equations in this paper were copied from the original paper exactly as they appeared.

However, the French text used rather archaic terms for a few concepts, which seriously interfered with the readability of the paper, at least for this reader. Where the meaning was obvious and unambiguous, and the term used was likely to be unfamiliar to many readers, I took the liberty of substituting the modern English term in the translation. Where the original usage seemed unlikely to cause confusion I translated it literally. Here is a brief list of some of the terms concerned:

- "force vives" -- literally, "lively forces". I translated this to "energy" or "kinetic energy" depending on the context. It is an old term for kinetic energy, as I found in perusing an unabridged dictionary.
- "quantité de movement" -- literally, "quantity of movement". I translated this as "momentum" everywhere, as that appears to be what Poincaré meant by it.
- "principe des aires" -- literally, "principle of areas". This is used in only one place. It is an old term for the principle of conservation of angular momentum – or, rather, the principle of areas is a subset of the law of conservation of angular momentum, as the latter implies the former but not conversely. I translated this as "conservation of angular momentum" but noted the literal translation as well in the text.
- "principe de réaction" -- literally, "principle of reaction". This can refer to Newton's third law – for every action there is an equal and opposite reaction – but Poincaré also uses it more generally, to refer to conservation of momentum. I think the meaning is generally clear from the context, and I just translated it literally.
- "principe de l'énergie" -- literally, "principle of energy". This refers to conservation of energy. I translated it literally in each case, as I think the meaning is generally clear from the context.
- "principe du movement relatif" -- literally, "principle of relative motion". I generally translated this literally, as it means just what it says: the principle of relativity, as applied to motion.

-- Steve Lawrence
January, 2008

Updated Versions

30 Jan 2008 – Minor corrections: removed doubled words, replaced use of "null" with "zero"